

CONSTRUCTING HYPER-IDEALS OF MULTILINEAR OPERATORS BETWEEN BANACH SPACES

Geraldo Botelho* and Ewerton R. Torres

Abstract

In view of the fact that some classical methods to construct multi-ideals fail in constructing hyper-ideals, in this paper we develop two new methods to construct hyper-ideals of multilinear operators between Banach spaces. These methods generate new classes of multilinear operators and show that some important well studied classes are Banach or p -Banach hyper-ideals.

1 Introduction and background

Ideals of multilinear operators between Banach spaces (or simply multi-ideals), which happen to be classes of multilinear operators that are stable with respect to the composition with linear operators, were introduced by Pietsch [23] as a first attempt to extend the successful theory of ideals of linear operators (operator ideals) to the nonlinear setting. The theory of multi-ideals turned out to be successful itself, and a refinement of this concept, called hyper-ideals, was introduced in [9] according to the following philosophy: the non-linearity of the multilinear setting is better explored by considering classes of multilinear operators that are stable with respect to the composition with multilinear operators – whenever this composition is possible, of course – rather than with linear operators.

The basics of the theory of hyper-ideals and plenty of distinguished examples can be found in [9]. As the theory of hyper-ideals is quite more restrictive than the theory of multi-ideals, it is expected that some techniques do not pass from multi-ideals to hyper-ideals. This is exactly what happens with some general methods to construct multi-ideals. While the technique concerning composition ideals works nicely for hyper-ideals (see [9, Theorem 4.2]), the factorization and the linearization methods are helpless in the realm of hyper-ideals (for a description of such methods, see, e.g. [5, 10]). An illustration of the failure of these methods in the generation of hyper-ideals can be found in Example 1.1. The purpose of this paper is to fill this gap by developing two general methods to construct hyper-ideals.

In Section 2 we introduce a method based on the transformation of finite vector-valued sequences by multilinear operators. We show that this method, which is akin to

*Supported by CNPq Grant 305958/2014-3 and Fapemig Grant PPM-00326-13.

2010 Mathematics Subject Classification: 47L20, 47B10, 46G25, 47L22.

Keywords: Banach spaces, multilinear operators, hyper-ideals, \mathcal{I} -bounded sets.

different sorts of *summing multilinear operators*, gives rise to new classes and recovers, as a particular instance, the important class of strongly summing multilinear operators. In Section 3 we show that, proceeding for multilinear operators as Aron and Rueda [2] did for homogeneous polynomials, we end up with Banach hyper-ideals. In this fashion, classical multi-ideals, such as compact, weakly compact and p -compact multilinear operators, are shown to be Banach hyper-ideals.

Along the paper, n is a positive integer, E, E_n, F, G, G_n, H , shall be Banach spaces over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , B_E denotes the closed unit ball of E , E' denotes the topological dual of E and $\mathcal{L}(E_1, \dots, E_n; F)$ stands for the Banach space of continuous n -linear operators from $E_1 \times \dots \times E_n$ to F endowed with the usual sup norm. When $F = \mathbb{K}$, the space of continuous n -linear forms is denoted by $\mathcal{L}(E_1, \dots, E_n)$. To avoid ambiguity, the space of n -linear forms on \mathbb{K} shall be denoted by $\mathcal{L}(^n\mathbb{K}; \mathbb{K})$. Given functionals $\varphi_1 \in E'_1, \dots, \varphi_n \in E'_n$ and a vector $b \in F$, $\varphi_1 \otimes \dots \otimes \varphi_n \otimes b$ denotes the element of $\mathcal{L}(E_1, \dots, E_n; F)$ given by

$$\varphi_1 \otimes \dots \otimes \varphi_n \otimes b(x_1, \dots, x_n) = \varphi_1(x_1) \dots \varphi_n(x_n)b.$$

The elements of the subspace of $\mathcal{L}(E_1, \dots, E_n; F)$ generated by operators of the form $\varphi_1 \otimes \dots \otimes \varphi_n \otimes b$ are called multilinear operators of finite type. A linear space-valued map is of finite rank if its range generates a finite dimensional subspace of the target space.

Normed, p -normed and Banach ideals of linear operators (operator ideals) are always meant in the sense of [13, 22]. According to [9], a *hyper-ideal of multilinear operators*, or simply a *hyper-ideal*, is a subclass \mathcal{H} of the class of all continuous multilinear operators between Banach spaces such that all components

$$\mathcal{H}(E_1, \dots, E_n; F) := \mathcal{L}(E_1, \dots, E_n; F) \cap \mathcal{H}$$

satisfy:

- (1) $\mathcal{H}(E_1, \dots, E_n; F)$ is a linear subspace of $\mathcal{L}(E_1, \dots, E_n; F)$ which contains the n -linear operators of finite type;
- (2) **The hyper-ideal property:** Given natural numbers n and $1 \leq m_1 < \dots < m_n$, and Banach spaces $G_1, \dots, G_{m_n}, E_1, \dots, E_n, F$ and H , if $B_1 \in \mathcal{L}(G_1, \dots, G_{m_1}; E_1), \dots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \dots, G_{m_n}; E_n)$, $t \in \mathcal{L}(F; H)$ and $A \in \mathcal{H}(E_1, \dots, E_n; F)$, then $t \circ A \circ (B_1, \dots, B_n)$ belongs to $\mathcal{H}(G_1, \dots, G_{m_n}; H)$.

If there exist $p \in (0, 1]$ and a map $\|\cdot\|_{\mathcal{H}}: \mathcal{H} \rightarrow [0, \infty)$ such that:

- (a) $\|\cdot\|_{\mathcal{H}}$ restricted to any component $\mathcal{H}(E_1, \dots, E_n; F)$ is a p -norm;
- (b) $\|I_n: \mathbb{K}^n \rightarrow \mathbb{K}, I_n(\lambda_1, \dots, \lambda_n) = \lambda_1 \dots \lambda_n\|_{\mathcal{H}} = 1$ for every n ;
- (c) **The hyper-ideal inequality:** If $A \in \mathcal{H}(E_1, \dots, E_n; F)$, $B_1 \in \mathcal{L}(G_1, \dots, G_{m_1}; E_1), \dots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \dots, G_{m_n}; E_n)$ and $t \in \mathcal{L}(F; H)$, then

$$\|t \circ A \circ (B_1, \dots, B_n)\|_{\mathcal{H}} \leq \|t\| \cdot \|A\|_{\mathcal{H}} \cdot \|B_1\| \dots \|B_n\|, \quad (1)$$

then $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is called a p -normed hyper-ideal. Normed, Banach and p -Banach hyper ideals are defined in the obvious way.

If the hyper-ideal property and the hyper-ideal inequality are required to hold only for the composition with linear operators on the left hand side, that is, if they hold in the particular case where $m_1 = 1, m_2 = 2, \dots, m_n = n$ and B_1, B_2, \dots, B_n are linear operators,

then we recover the notion of multi-ideals (normed, p -normed, Banach, p -Banach multi-ideals). For the theory of multi-ideals we refer to [5, 16, 20].

Let us give an illustrative example of the failure of the factorization method in the generation of hyper-ideals:

Example 1.1. Starting the factorization method with the ideal π_p of absolutely p -summing operators, we obtain the multi-ideal $\mathcal{L}_{d,p}$ of p -dominated multilinear operators [23, Theorem 13] (for a proof, see [8, Remark 3.3]). In [26, Proposition 4.2(b)] it is proved that $\mathcal{L}_{d,p}$ fails to be a hyper-ideal.

The following criterion shall be used twice:

Theorem 1.2. [9, Theorem 2.5] *A class \mathcal{H} of continuous multilinear operators endowed with a map $\|\cdot\|_{\mathcal{H}}: \mathcal{H} \rightarrow [0, +\infty)$ is a p -Banach hyper-ideal, $0 < p \leq 1$, if and only if the following conditions are satisfied:*

- (i) $I_n \in \mathcal{H}(^n \mathbb{K}; \mathbb{K})$ and $\|I_n\|_{\mathcal{H}} = 1$ for every $n \in \mathbb{N}$;
- (ii) If $(A_j)_{j=1}^{\infty} \subseteq \mathcal{H}(E_1, \dots, E_n; F)$ is such that $\sum_{j=1}^{\infty} \|A_j\|_{\mathcal{H}}^p < \infty$, then

$$A := \sum_{j=1}^{\infty} A_j \in \mathcal{H}(E_1, \dots, E_n; F) \text{ and } \|A\|_{\mathcal{H}}^p \leq \sum_{j=1}^{\infty} \|A_j\|_{\mathcal{H}}^p;$$

- (iii) $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ enjoys the hyper-ideal property and the hyper-ideal inequality.

Remark 1.3. In [9, Corollary 3.3] it is proved that every hyper-ideal contains the finite rank multilinear operators.

2 The inequality method

The method that we introduce in this section generates new hyper-ideals, such as the class of strongly almost summing operators (cf. Corollary 2.11), and recovers, as particular instances, well studied classes, such as strongly summing multilinear operators (cf. Corollary 2.8).

Definition 2.1. (a) Let $0 < p \leq 1$. By \mathcal{BAN} we denote the class of all Banach spaces over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and by $p\text{-}\mathcal{BAN}$ the class of all p -Banach spaces over \mathbb{K} . A correspondence

$$\mathcal{X}: \mathcal{BAN} \longrightarrow p\text{-}\mathcal{BAN}$$

that associates to each Banach space E a p -Banach space $(\mathcal{X}(E), \|\cdot\|_{\mathcal{X}(E)})$ is called a *p -sequence functor* if:

- (i) $\mathcal{X}(E)$ is a linear subspace of $E^{\mathbb{N}}$ with the usual algebraic operations;
- (ii) For all $x \in E$ and $j \in \mathbb{N}$, we have $(0, \dots, 0, x, 0, \dots) \in \mathcal{X}(E)$, where x is placed at the j -th coordinate, and $\|(0, \dots, 0, x, 0, \dots)\|_{\mathcal{X}(E)} = \|x\|_E$.
- (iii) For every $u \in \mathcal{L}(E; F)$ and every finite E -valued sequence $(x_j)_{j=1}^k := (x_1, \dots, x_k, 0, 0, \dots)$, $k \in \mathbb{N}$, it holds

$$\|(u(x_j))_{j=1}^k\|_{\mathcal{X}(F)} \leq \|u\| \cdot \|(x_j)_{j=1}^k\|_{\mathcal{X}(E)}.$$

When $p = 1$ we simply say that \mathcal{X} is a *sequence functor*.

(b) Let $0 < p, q \leq 1$. A p -sequence functor \mathcal{X} is *scalarly dominated* by the q -sequence functor \mathcal{Y} if, for every finite sequence $(\lambda_j)_{j=1}^k \subseteq \mathbb{K}$, $k \in \mathbb{N}$, we have

$$\|(\lambda_j)_{j=1}^k\|_{\mathcal{X}(\mathbb{K})} \leq \|(\lambda_j)_{j=1}^k\|_{\mathcal{Y}(\mathbb{K})}.$$

The term *sequence functor* was used in [7] in a different sense.

Example 2.2. (a) For $p > 0$, the following correspondences are p -sequence functors (sequence functors if $p \geq 1$):

- $E \mapsto (\ell_p(E), \|\cdot\|_p)$ (absolutely p -summable sequences).
- $E \mapsto (\ell_p^w(E), \|\cdot\|_{w,p})$ (weakly p -summable sequences [14]).
- $E \mapsto (\ell_p^u(E), \|\cdot\|_{u,p})$ (unconditionally p -summable sequences [13]).
- $E \mapsto (\ell_p\langle E \rangle, \|\cdot\|_{\ell_p\langle E \rangle})$ (Cohen strongly p -summable sequences [11]).

(b) The following correspondences are sequence functors:

- $E \mapsto (c_0(E), \|\cdot\|_\infty)$ (norm null sequences).
- $E \mapsto (c_0^w(E), \|\cdot\|_\infty)$ (weakly null sequences).
- $E \mapsto (\ell_\infty(E), \|\cdot\|_\infty)$ (bounded sequences).
- $E \mapsto (Rad(E), \|\cdot\|_{Rad(E)})$, where $Rad(E)$ is the space of almost unconditionally summable E -valued sequences [14, Chapter 12] and

$$\|(x_j)_{j=1}^\infty\|_{Rad(E)} = \left(\int_0^1 \left\| \sum_{j=1}^\infty r_j(t)x_j \right\|^2 dt \right)^{1/2},$$

where $(r_j)_{j=1}^\infty$ are the Rademacher functions.

(c) Among several other obvious dominations, for $0 < p < q$, $\ell_q(\cdot)$ is scalarly dominated by $\ell_p(\cdot)$, as well as $\ell_q^w(\cdot)$ by $\ell_p^w(\cdot)$, $\ell_q^u(\cdot)$ by $\ell_p^u(\cdot)$ and $\ell_q\langle \cdot \rangle$ by $\ell_p\langle \cdot \rangle$.

To introduce the inequality method we need the

Lemma 2.3. *Let \mathcal{X} be a p -sequence functor, $0 < p \leq 1$ and $(x_j^1)_{j=1}^k \subseteq E_1, \dots, (x_j^n)_{j=1}^k \subseteq E_n$ be finite sequences. Then*

$$\sup_{T \in B_{\mathcal{L}(E_1, \dots, E_n)}} \|(T(x_j^1, \dots, x_j^n))_{j=1}^k\|_{\mathcal{X}(\mathbb{K})} < \infty.$$

Proof. For any n -linear form $T \in B_{\mathcal{L}(E_1, \dots, E_n)}$,

$$\begin{aligned} \|(T(x_j^1, \dots, x_j^n))_{j=1}^k\|_{\mathcal{X}(\mathbb{K})}^p &= \|(T(x_1^1, \dots, x_1^n), 0, \dots, 0, \dots) + (0, T(x_2^1, \dots, x_2^n), 0, \dots, 0, \dots) \\ &\quad + \dots + (0, 0, \dots, 0, T(x_k^1, \dots, x_k^n), 0, \dots)\|_{\mathcal{X}(\mathbb{K})}^p \\ &\leq \|(T(x_1^1, \dots, x_1^n), 0, \dots, 0, \dots)\|_{\mathcal{X}(\mathbb{K})}^p + \|(0, T(x_2^1, \dots, x_2^n), 0, \dots, 0, \dots)\|_{\mathcal{X}(\mathbb{K})}^p \\ &\quad + \dots + \|(0, 0, \dots, 0, T(x_k^1, \dots, x_k^n), 0, \dots)\|_{\mathcal{X}(\mathbb{K})}^p \\ &= |T(x_1^1, \dots, x_1^n)|^p + |T(x_2^1, \dots, x_2^n)|^p + \dots + |T(x_k^1, \dots, x_k^n)|^p \\ &\leq \|x_1^1\|^p \dots \|x_1^n\|^p + \|x_2^1\|^p \dots \|x_2^n\|^p + \dots + \|x_k^1\|^p \dots \|x_k^n\|^p. \end{aligned}$$

The result follows because the latter term does not depend on T . □

Definition 2.4. Let $0 < p, q \leq 1$, \mathcal{X} be a p -sequence functor and \mathcal{Y} be a q -sequence functor. An n -linear operator $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be $(\mathcal{X}; \mathcal{Y})$ -*summing* if there is a constant $C > 0$ such that

$$\|(A(x_j^1, \dots, x_j^n))_{j=1}^k\|_{\mathcal{Y}(F)} \leq C \cdot \sup_{T \in B_{\mathcal{L}(E_1, \dots, E_n)}} \|(T(x_j^1, \dots, x_j^n))_{j=1}^k\|_{\mathcal{X}(\mathbb{K})}, \quad (2)$$

for every $k \in \mathbb{N}$ and all finite sequences $(x_j^1)_{j=1}^k \subseteq E_1, \dots, (x_j^n)_{j=1}^k \subseteq E_n$. In this case we write $A \in (\mathcal{X}; \mathcal{Y})(E_1, \dots, E_n; F)$ and define

$$\|A\|_{(\mathcal{X}; \mathcal{Y})} = \inf\{C > 0 : C \text{ satisfies (2)}\}.$$

To prove that the inequality method generates hyper-ideals we need the following result. The proof is standard and we omit it.

Lemma 2.5. Let \mathcal{X} be a p -sequence functor, $0 < p \leq 1$, and $n \in \mathbb{N}$. If $\sum_{j=1}^{\infty} x_j^1, \dots, \sum_{j=1}^{\infty} x_j^n$ are convergent series in E , then the series $\sum_{j=1}^{\infty} (x_j^1, \dots, x_j^n, 0, 0, \dots)$ converges in $\mathcal{X}(E)$ and

$$\sum_{j=1}^{\infty} (x_j^1, \dots, x_j^n, 0, 0, \dots) = \left(\sum_{j=1}^{\infty} x_j^1, \dots, \sum_{j=1}^{\infty} x_j^n, 0, 0, \dots \right) \text{ in } \mathcal{X}(E).$$

Theorem 2.6. Let $0 < p, q \leq 1$ and \mathcal{Y} be a q -sequence functor scalarly dominated by the p -sequence functor \mathcal{X} . Then $((\mathcal{X}; \mathcal{Y}), \|\cdot\|_{(\mathcal{X}; \mathcal{Y})})$ is a q -Banach hyper-ideal.

Proof. We prove that the conditions of Theorem 1.2 are fulfilled.

- (i) It is routine to prove that, for each $n \in \mathbb{N}$, $I_n \in (\mathcal{X}; \mathcal{Y})({}^n\mathbb{K}; \mathbb{K})$. As $I_n \in B_{\mathcal{L}({}^n\mathbb{K}; \mathbb{K})}$ and \mathcal{Y} is scalarly dominated by \mathcal{X} , we get $\|I_n\|_{(\mathcal{X}; \mathcal{Y})} \leq 1$. Let C be a constant working in (2) for I_n . Choosing $k = x_1^1, \dots, x_n^1 = 1$ we obtain $C \geq 1$, from which follows $\|I_n\|_{(\mathcal{X}; \mathcal{Y})} = 1$.
- (ii) Let $(A_i)_{i=1}^{\infty} \subseteq (\mathcal{X}; \mathcal{Y})(E_1, \dots, E_n; F)$ be such that $\sum_{i=1}^{\infty} \|A_i\|_{(\mathcal{X}; \mathcal{Y})}^q < \infty$. We are supposed to show that

$$A := \sum_{i=1}^{\infty} A_i \in (\mathcal{X}; \mathcal{Y})(E_1, \dots, E_n; F) \text{ and } \|A\|_{(\mathcal{X}; \mathcal{Y})}^q \leq \sum_{i=1}^{\infty} \|A_i\|_{(\mathcal{X}; \mathcal{Y})}^q. \quad (3)$$

First note that, for each operator $B \in (\mathcal{X}; \mathcal{Y})(E_1, \dots, E_n; F)$ and all $x_j \in E_j$, $j = 1, \dots, n$, we have

$$\begin{aligned} \|B(x_1, \dots, x_n)\|_F &= \|(B(x_1, \dots, x_n), 0, \dots, 0, \dots)\|_{\mathcal{Y}(F)} \\ &\leq C \cdot \sup_{T \in B_{\mathcal{L}(E_1, \dots, E_n)}} \|(T(x_1, \dots, x_n), 0, \dots, 0, \dots)\|_{\mathcal{X}(\mathbb{K})} \\ &= C \cdot \sup_{T \in B_{\mathcal{L}(E_1, \dots, E_n)}} |T(x_1, \dots, x_n)| \leq C \cdot \|x_1\| \cdots \|x_n\|, \end{aligned}$$

for each constant C working in (2) for B . It follows that $\|B\| \leq \|B\|_{(\mathcal{X}; \mathcal{Y})}$. Hence, as $q \leq 1$, the series $\sum_{i=1}^{\infty} A_i$ is absolutely convergent in the Banach space $\mathcal{L}(E_1, \dots, E_n; F)$, therefore

convergent, say $A := \sum_{i=1}^{\infty} A_i \in \mathcal{L}(E_1, \dots, E_n; F)$. Given $k \in \mathbb{N}$ and finite sequences $(x_j^1)_{j=1}^k \subseteq E_1, \dots, (x_j^n)_{j=1}^k \subseteq E_n$, Lemma 2.5 gives

$$\begin{aligned} \|(A(x_j^1, \dots, x_j^n))_{j=1}^k\|_{\mathcal{Y}(F)}^q &= \left\| \left(\sum_{i=1}^{\infty} A_i(x_j^1, \dots, x_j^n) \right)_{j=1}^k \right\|_{\mathcal{Y}(F)}^q \\ &= \left\| \sum_{i=1}^{\infty} (A_i(x_j^1, \dots, x_j^n))_{j=1}^k \right\|_{\mathcal{Y}(F)}^q \leq \sum_{i=1}^{\infty} \|(A_i(x_j^1, \dots, x_j^n))_{j=1}^k\|_{\mathcal{Y}(F)}^q \\ &= \left(\sum_{i=1}^{\infty} \|A_i\|_{(\mathcal{X}; \mathcal{Y})}^q \right) \cdot \left(\sup_{T \in B_{\mathcal{L}(E_1, \dots, E_n)}} \|(T(x_j^1, \dots, x_j^n))_{j=1}^k\|_{\mathcal{X}(\mathbb{K})} \right)^q, \end{aligned}$$

proving (3).

(iii) Let $\leq m_1 < \dots < m_n$ be natural numbers, $A \in (\mathcal{X}; \mathcal{Y})(E_1, \dots, E_n; F)$, $B_1 \in \mathcal{L}(G_1, \dots, G_{m_1}; E_1)$, \dots , $B_n \in \mathcal{L}(G_{m_{n-1}+1}, \dots, G_{m_n}; E_n)$ and $t \in \mathcal{L}(F; H)$. Of course we can assume that $B_l \neq 0$ for $l = 1, \dots, n$. For every $T \in B_{\mathcal{L}(E_1, \dots, E_n)}$,

$$T \circ \left(\frac{B_1}{\|B_1\|}, \dots, \frac{B_n}{\|B_n\|} \right) \in B_{\mathcal{L}(G_1, \dots, G_{m_n})},$$

thus

$$\begin{aligned} &\|(t \circ A \circ (B_1, \dots, B_n)(x_j^1, \dots, x_j^n))_{j=1}^k\|_{\mathcal{Y}(H)} \\ &\leq \|t\| \cdot \|(A(B_1(x_j^1, \dots, x_j^{m_1}), \dots, B_n(x_j^{m_{n-1}+1}, \dots, x_j^{m_n})))_{j=1}^k\|_{\mathcal{Y}(F)} \\ &\leq \|t\| \cdot \|A\|_{(\mathcal{X}; \mathcal{Y})} \cdot \sup_{T \in B_{\mathcal{L}(E_1, \dots, E_n)}} \|(T(B_1(x_j^1, \dots, x_j^{m_1}), \dots, B_n(x_j^{m_{n-1}+1}, \dots, x_j^{m_n})))_{j=1}^k\|_{\mathcal{X}(\mathbb{K})} \\ &= \|t\| \cdot \|A\|_{(\mathcal{X}; \mathcal{Y})} \cdot \|B_1\| \cdots \|B_n\| \cdot \\ &\quad \sup_{T \in B_{\mathcal{L}(E_1, \dots, E_n)}} \left\| \left(T \left(\frac{B_1}{\|B_1\|}(x_j^1, \dots, x_j^{m_1}), \dots, \frac{B_n}{\|B_n\|}(x_j^{m_{n-1}+1}, \dots, x_j^{m_n}) \right) \right)_{j=1}^k \right\|_{\mathcal{X}(\mathbb{K})} \\ &\leq \|t\| \cdot \|A\|_{(\mathcal{X}; \mathcal{Y})} \cdot \|B_1\| \cdots \|B_n\| \cdot \sup_{S \in B_{\mathcal{L}(G_1, \dots, G_{m_n})}} \|(S(x_j^1, \dots, x_j^{m_n}))_{j=1}^k\|_{\mathcal{X}(\mathbb{K})}, \end{aligned}$$

where the first inequality follows from condition 2.1(iii). It follows that $t \circ A \circ (B_1, \dots, B_n) \in (\mathcal{X}; \mathcal{Y})(G_1, \dots, G_{m_n}; H)$ and

$$\|t \circ A \circ (B_1, \dots, B_n)\|_{(\mathcal{X}; \mathcal{Y})} \leq \|t\| \cdot \|A\|_{(\mathcal{X}; \mathcal{Y})} \cdot \|B_1\| \cdots \|B_n\|.$$

□

Next we provide two examples of hyper-ideals generated by the inequality method.

In the first example we show that the inequality method recovers an important well studied class as a particular instance. The class of dominated multilinear operators was introduced by Pietsch [23] as a first attempt to generalize the classical ideal of absolutely summing linear operators to the multilinear setting. Although several other classes of

absolutely summing multilinear operators have appeared, the class of dominated multilinear operators keeps being studied to this day. Among other recent developments, Popa [26] proved that the class of dominated multilinear operators fails to be a hyper-ideal. It is a natural question to ask if there is room in the realm of Banach hyper-ideals for a multilinear generalization of the Banach ideal of absolutely p -summing linear operators. We found the answer in the following class introduced by Dimant [15]:

Definition 2.7. For $0 < p < \infty$, an n -linear operator $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is *strongly p -summing* if there is $C > 0$ such that

$$\left(\sum_{i=1}^k \|A(x_i^1, \dots, x_i^n)\|^p \right)^{1/p} \leq C \cdot \sup_{T \in B_{\mathcal{L}(E_1, \dots, E_n)}} \left(\sum_{i=1}^k |T(x_i^1, \dots, x_i^n)|^p \right)^{1/p} \quad (4)$$

for all $(x_i^l)_{i=1}^k \subseteq E_l$, $l = 1, \dots, n$, $k \in \mathbb{N}$. In this case we write $A \in \mathcal{L}_{ss}^p(E_1, \dots, E_n; F)$ and define

$$\|A\|_{ss,p} = \inf\{C > 0 : C \text{ satisfies (4)}\}.$$

It is clear that the linear component of \mathcal{L}_{ss}^p recovers the ideal of absolutely p -summing linear operators. Making $\mathcal{Y}(E) = \mathcal{X}(E) = \ell_p(E)$ for every Banach space E in Theorem 2.6, we obtain:

Corollary 2.8. *The class $(\mathcal{L}_{ss}^p, \|\cdot\|_{ss,p})$ of strongly p -summing multilinear operators is a p -Banach hyper-ideal for $0 < p < 1$ and a Banach hyper-ideal for $p \geq 1$.*

The second example of a hyper-ideal generated by the inequality method is a new class. The ideal of almost summing linear operators was introduced in [14] and several classes of *almost summing multilinear operators* have been studied, see, e.g. [4, 6, 19, 21, 25]. Such classes often fail to be hyper-ideals:

Example 2.9. Let $\mathcal{L}_{al.s}$ denote the class of almost summing multilinear operators introduced in [4, 6]. By [4, Example 4.3(2)], there are multilinear operators of finite rank not belonging to $\mathcal{L}_{al.s}$, so $\mathcal{L}_{al.s}$ fails to be a hyper-ideal by Remark 1.3.

Our second example is a hyper-ideal generated by the inequality method that generalizes the ideal of almost summing linear operators to the multilinear setting. Remember that $(Rad(\cdot), \|\cdot\|_{Rad(E)})$ denotes the sequence functor of almost unconditionally summable sequences (cf. Example 2.2(b)).

Definition 2.10. Let $p \geq 1$. We say that an n -linear operator $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is *strongly almost p -summing* if there is a constant $C > 0$ such that, for all $k \in \mathbb{N}$ and $(x_j^1)_{j=1}^k \subseteq E_1, \dots, (x_j^n)_{j=1}^k \subseteq E_n$,

$$\|(A(x_j^1, \dots, x_j^n))_{j=1}^k\|_{Rad(F)} \leq C \cdot \sup_{T \in B_{\mathcal{L}(E_1, \dots, E_n)}} \left(\sum_{j=1}^k |(T(x_j^1, \dots, x_j^n))_{j=1}^k|^p \right)^{1/p}. \quad (5)$$

In this case we write $A \in \mathcal{L}_{sas,p}(E_1, \dots, E_n; F)$ and define

$$\|A\|_{\mathcal{L}_{sas,p}} = \inf\{C > 0 : C \text{ satisfies (5)}\}.$$

Corollary 2.11. *If $0 < p \leq 2$, then $(\mathcal{L}_{sas,p}, \|\cdot\|_{\mathcal{L}_{sas,p}})$ is an Banach hyper-ideal.*

Proof. Note that $\mathcal{L}_{sas,p}$ is precisely the class of $(\ell_p(\cdot); \text{Rad}(\cdot))$ -summing multilinear operators. As $\ell_p(\cdot)$ is a p -sequence functor, $\text{Rad}(\cdot)$ is a sequence functor and $\text{Rad}(\cdot)$ is scalarly dominated by $\ell_p(\cdot)$ (remember that $\text{Rad}(\mathbb{K}) = \ell_2$ isometrically and $0 < p \leq 2$), the result follows from Theorem 2.6. \square

Remark 2.12. A method to generate multi-ideals (not hyper-ideals), related to the method introduced in this section, is presented in [27]. However, an important assumption is missing there. More precisely, for Theorem [27, Theorem 3] to be true, the linear stability of the underlying sequence spaces is required, that is, condition 2.1(iii) must be added to the assumptions of [27, Theorem 3].

3 The boundedness method

As mentioned in the Introduction, some methods of generating multi-ideals starting with a given operator ideal are not effective to generate hyper-ideals (cf. Example 1.1). In this section, inspired by the polynomial case studied in [2], we introduce a method to generate hyper-ideals starting with a given operator ideal.

The notion of \mathcal{I} -bounded sets, where \mathcal{I} is an operator ideal, was introduced by Stephani [29]; recent developments can be found, e.g., in [2, 3, 17, 18].

Definition 3.1. Let \mathcal{I} be an operator ideal. A subset K of a Banach space F is said to be \mathcal{I} -bounded if there are a Banach space H and an operator $u \in \mathcal{I}(H; F)$ such that $K \subseteq u(B_H)$. The collection of all \mathcal{I} -bounded subsets of F is denoted by $C_{\mathcal{I}}(F)$.

Aron and Rueda [2] used the concept of \mathcal{I} -bounded set to define an ideal of homogeneous polynomials. In this section we show that, proceeding for multilinear operators as Aron and Rueda proceeded for polynomials, we end up with a Banach hyper-ideal.

Definition 3.2. Let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be a p -normed operator ideal, $0 < p \leq 1$. We say that a multilinear operator $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is \mathcal{I} -bounded if $A(B_{E_1} \times \dots \times B_{E_n}) \in C_{\mathcal{I}}(F)$, that is, if there are a Banach space H and an operator $u \in \mathcal{I}(H; F)$ such that

$$A(B_{E_1} \times \dots \times B_{E_n}) \subseteq u(B_H). \quad (6)$$

In this case we write $A \in \mathcal{L}_{(\mathcal{I})}(E_1, \dots, E_n; F)$ and define

$$\|A\|_{\mathcal{L}_{(\mathcal{I})}} = \inf\{\|u\|_{\mathcal{I}} : u \text{ satisfies (6)}\}.$$

Theorem 3.3. *Let $0 < p \leq 1$ and $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be a p -normed (p -Banach) operator ideal. Then $(\mathcal{L}_{(\mathcal{I})}, \|\cdot\|_{\mathcal{L}_{(\mathcal{I})}})$ is a p -normed (p -Banach) hyper-ideal.*

Proof. We omit the proof of the incomplete case. Let us apply Theorem 1.2 to prove that $(\mathcal{L}_{(\mathcal{I})}, \|\cdot\|_{\mathcal{L}_{(\mathcal{I})}})$ is a p -Banach hyper-ideal whenever $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a p -Banach operator ideal.
(i) As $\text{Id}_{\mathbb{K}} \in \mathcal{I}(\mathbb{K}; \mathbb{K})$ and $\|\text{Id}_{\mathbb{K}}\|_{\mathcal{I}} = 1$, it follows easily that $I_n \in \mathcal{L}_{(\mathcal{I})}(^n \mathbb{K}; \mathbb{K})$ and

$\|I_n\|_{\mathcal{L}(\mathcal{I})} \leq 1$. For all H and $u \in \mathcal{I}(H; \mathbb{K})$ such that $I_n((B_{\mathbb{K}})^n) \subseteq u(B_H)$, choosing $z \in B_H$ such that $u(z) = 1$ we get

$$\|u\|_{\mathcal{I}} \geq \|u\| \geq |u(z)| = 1,$$

from which we conclude that $\|I_n\|_{\mathcal{L}(\mathcal{I})} = 1$.

(ii) Let $(A_j)_{j=1}^{\infty} \subseteq \mathcal{L}(\mathcal{I})(E_1, \dots, E_n; F)$ be such that $\sum_{j=1}^{\infty} \|A_j\|_{\mathcal{L}(\mathcal{I})}^p < \infty$. Let $\varepsilon > 0$. For each $j \in \mathbb{N}$, the set $A_j(B_{E_1} \times \dots \times B_{E_n})$ is \mathcal{I} -bounded, thus there exist a Banach space H_j and an operator $u_j \in \mathcal{I}(H_j; F)$ such that

$$A_j(B_{E_1} \times \dots \times B_{E_n}) \subseteq u_j(B_{H_j})$$

and $\|u_j\|_{\mathcal{I}} < (1 + \varepsilon)\|A_j\|_{\mathcal{L}(\mathcal{I})}$. So,

$$\|A_j\| \leq \|u_j\| \leq \|u_j\|_{\mathcal{I}} < (1 + \varepsilon)\|A_j\|_{\mathcal{L}(\mathcal{I})}.$$

Making $\varepsilon \rightarrow 0$ we obtain $\|A_j\| \leq \|A_j\|_{\mathcal{L}(\mathcal{I})}$. As $p \leq 1$, we conclude that the series $\sum_{j=1}^{\infty} A_j$ is absolutely convergent in the Banach space $\mathcal{L}(E_1, \dots, E_n; F)$, hence convergent,

say $A := \sum_{j=1}^{\infty} A_j \in \mathcal{L}(E_1, \dots, E_n; F)$. Let

$$H := \left(\bigoplus_{j=1}^{\infty} H_j, \|\cdot\|_{\infty} \right)$$

be the Banach space of bounded sequences $(x_j)_{j=1}^{\infty}$, where $x_j \in H_j$ for every j , endowed with the sup norm. Letting $\pi_j: H \rightarrow H_j$, $j \in \mathbb{N}$, be the canonical projections and defining

$$v_j: H \rightarrow F, \quad v_j = u_j \circ \pi_j,$$

by the ideal property of \mathcal{I} we have that each $v_j \in \mathcal{I}(H; F)$ and

$$\sum_{j=1}^{\infty} \|v_j\|_{\mathcal{I}}^p = \sum_{j=1}^{\infty} \|u_j \circ \pi_j\|_{\mathcal{I}}^p \leq \sum_{j=1}^{\infty} \|u_j\|_{\mathcal{I}}^p < (1 + \varepsilon)^p \cdot \sum_{j=1}^{\infty} \|A_j\|_{\mathcal{L}(\mathcal{I})}^p < \infty.$$

Since $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a p -Banach ideal, it follows that

$$u := \sum_{j=1}^{\infty} v_j \in \mathcal{I}(H; F) \quad \text{and} \quad \|u\|_{\mathcal{I}}^p < (1 + \varepsilon)^p \cdot \sum_{j=1}^{\infty} \|A_j\|_{\mathcal{L}(\mathcal{I})}^p.$$

Given $y \in A(B_{E_1} \times \dots \times B_{E_n})$, choose $x_l \in B_{E_l}$, $l = 1, \dots, n$, such that

$$y = A(x_1, \dots, x_n) = \sum_{j=1}^{\infty} A_j(x_1, \dots, x_n).$$

As $A_j(B_{E_1} \times \dots \times B_{E_n}) \subseteq u_j(B_{H_j})$, for every j there is $z_j \in B_{H_j}$ such that $A_j(x_1, \dots, x_n) = u_j(z_j)$. Hence,

$$y = \sum_{j=1}^{\infty} A_j(x_1, \dots, x_n) = \sum_{j=1}^{\infty} u_j(z_j) = \sum_{j=1}^{\infty} u_j \circ \pi_j((z_k)_{k=1}^{\infty}) = \sum_{j=1}^{\infty} v_j((z_k)_{k=1}^{\infty}) = u((z_k)_{k=1}^{\infty}),$$

where $(z_k)_{k=1}^\infty \in B_H$ because $\|z_j\| \leq 1$ for all j . Then $y \in u(B_H)$, which means that $A(B_{E_1} \times \cdots \times B_{E_n}) \in C_{\mathcal{I}}(F)$. This proves that $A \in \mathcal{L}_{\langle \mathcal{I} \rangle}(E_1, \dots, E_n; F)$ and

$$\|A\|_{\mathcal{L}_{\langle \mathcal{I} \rangle}}^p \leq \|u\|_{\mathcal{I}}^p < (1 + \varepsilon)^p \cdot \sum_{j=1}^{\infty} \|A_j\|_{\mathcal{L}_{\langle \mathcal{I} \rangle}}^p.$$

Just make $\varepsilon \rightarrow 0$ to obtain $\|A\|_{\mathcal{L}_{\langle \mathcal{I} \rangle}}^p \leq \sum_{j=1}^{\infty} \|A_j\|_{\mathcal{L}_{\langle \mathcal{I} \rangle}}^p$.

(iii) Let $t \in \mathcal{L}(F; G)$, $A \in \mathcal{L}_{\langle \mathcal{I} \rangle}(E_1, \dots, E_n; F)$ and $B_1 \in \mathcal{L}(G_1, \dots, G_{m_1}; E_1), \dots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \dots, G_{m_n}; E_n)$, where $1 \leq m_1 < \cdots < m_n$. By the definition of $\mathcal{L}_{\langle \mathcal{I} \rangle}$ there are a Banach space H and an operator $u \in \mathcal{I}(H; F)$ such that

$$A(B_{E_1} \times \cdots \times B_{E_n}) \subseteq u(B_H). \quad (7)$$

Of course we can assume $B_l \neq 0$ for $l = 1, \dots, n$, and in this case

$$\frac{B_l}{\|B_l\|} (B_{G_{m_{l-1}+1}} \times \cdots \times B_{G_{m_l}}) \subseteq B_{E_l},$$

what gives

$$A \circ \left(\frac{B_1}{\|B_1\|}, \dots, \frac{B_n}{\|B_n\|} \right) (B_{G_1} \times \cdots \times B_{G_{m_n}}) \subseteq u(B_H).$$

Hence,

$$t \circ A \circ (B_1, \dots, B_n) (B_{G_1} \times \cdots \times B_{G_{m_n}}) \subseteq (\|B_1\| \cdots \|B_n\| t \circ u)(B_H).$$

This proves that

$$t \circ A \circ (B_1, \dots, B_n) \in \mathcal{L}_{\langle \mathcal{I} \rangle}(G_1, \dots, G_{m_n}; G),$$

because $\|B_1\| \cdots \|B_n\| t \circ u \in \mathcal{I}(H; G)$, and that

$$\|t \circ A \circ (B_1, \dots, B_n)\|_{\mathcal{L}_{\langle \mathcal{I} \rangle}} \leq \|t\| \cdot \|u\|_{\mathcal{I}} \cdot \|B_1\| \cdots \|B_n\|.$$

Taking the infimum over all operators u satisfying (7) we get the desired hyper-ideal inequality. \square

Example 3.4. Let \mathcal{K} and \mathcal{W} denote the ideals of compact and weakly compact linear operators. Reasoning as in [2, Example 3.1], we see that $C_{\mathcal{K}}(E)$ is the collection of relatively compact subsets of E and that $C_{\mathcal{W}}(E)$ is the collection of relatively weakly compact subsets of E . Then the classes $\mathcal{L}_{\mathcal{K}}$ of compact multilinear operators and $\mathcal{L}_{\mathcal{W}}$ of weakly compact multilinear operators are closed hyper-ideals (that is, Banach hyper-ideals with the usual sup norm) by Theorem 3.3

The information in the example above was obtained in [9] by a different reasoning. To give new applications of the method introduced in this section, we consider the following concept introduced by Sinha and Karn [28], which has been playing an important role in the theory of operator ideals (cf. [18, 24] and references therein) and in the study of variants of the approximation property (cf. [12, 18] and references therein).

Definition 3.5. Given $1 \leq p < \infty$, let p' be given by $\frac{1}{p} + \frac{1}{p'} = 1$. A subset K of a Banach space E is said to be *relatively p -compact* if there is a sequence $(x_j)_{j=1}^\infty \in \ell_p(E)$ such that

$$K \subseteq \left\{ \sum_{j=1}^\infty \lambda_j x_j : (\lambda_j)_{j=1}^\infty \in B_{\ell_{p'}} \right\}. \quad (8)$$

The definition below is the multilinear counterpart of the polynomial case studied by Aron and Rueda [1].

Definition 3.6. An n -linear operator $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be *p -compact*, $p \geq 1$, if $A(B_{E_1} \times \dots \times B_{E_n})$ is a relatively p -compact subset of F . In this case we write $A \in \mathcal{L}_{\mathcal{K}_p}(E_1, \dots, E_n; F)$ and define

$$\|A\|_{\mathcal{L}_{\mathcal{K}_p}} = \inf \left\{ \|(x_j)_{j=1}^\infty\|_p : (x_j)_{j=1}^\infty \text{ satisfies (8) for } A(B_{E_1} \times \dots \times B_{E_n}) \right\}.$$

The linear component of $\mathcal{L}_{\mathcal{K}_p}$ recovers the intensively studied ideal \mathcal{K}_p of p -compact linear operators.

Proposition 3.7. For every $p \geq 1$, $\mathcal{L}_{\mathcal{K}_p} = \mathcal{L}_{\langle \mathcal{K}_p \rangle}$ and $\|\cdot\|_{\mathcal{L}_{\mathcal{K}_p}} = \|\cdot\|_{\mathcal{L}_{\langle \mathcal{K}_p \rangle}}$. In particular, the class $\mathcal{L}_{\mathcal{K}_p}$ of p -compact multilinear operators is a Banach hyper-ideal.

Proof. In [2, Example 3.1] it is proved that, for every Banach space E , $C_{\mathcal{K}_p}(E)$ coincides with the collection of relatively p -compact subsets of E . So, $\mathcal{L}_{\mathcal{K}_p} = \mathcal{L}_{\langle \mathcal{K}_p \rangle}$. The inequality $\|\cdot\|_{\mathcal{L}_{\mathcal{K}_p}} \leq \|\cdot\|_{\mathcal{L}_{\langle \mathcal{K}_p \rangle}}$ follows easily from the definitions. Given $A \in \mathcal{L}_{\mathcal{K}_p}(E_1, \dots, E_n; F)$ and a sequence $(x_j)_{j=1}^\infty \in \ell_p(F)$ satisfying (8) for $A(B_{E_1} \times \dots \times B_{E_n})$, the linear operator

$$u: \ell_{p'} \longrightarrow F, \quad u((\lambda_j)_{j=1}^\infty) = \sum_{j=1}^\infty \lambda_j x_j,$$

is well defined by Hölder's inequality. In

$$A(B_{E_1} \times \dots \times B_{E_n}) \subseteq \left\{ \sum_{j=1}^\infty \lambda_j x_j : (\lambda_j)_{j=1}^\infty \in B_{\ell_{p'}} \right\} = u(B_{\ell_{p'}}),$$

the equality shows that u is a p -compact linear operator and the inclusion gives

$$\|A\|_{\mathcal{L}_{\langle \mathcal{K}_p \rangle}} \leq \|u\|_{\mathcal{K}_p} \leq \|(x_j)_{j=1}^\infty\|_p.$$

Taking the infimum over all such sequences $(x_j)_{j=1}^\infty$ we conclude that $\|A\|_{\mathcal{L}_{\langle \mathcal{K}_p \rangle}} \leq \|A\|_{\mathcal{L}_{\mathcal{K}_p}}$.
Now the second assertion follows from Theorem 3.3. \square

Acknowledgement. We thank Professors Mary Lilian Lourenço for making our collaboration possible and Vinícius V. Fávaro for his helpful suggestions.

References

- [1] R. M. Aron and P. Rueda, *p-Compact homogeneous polynomials from an ideal point of view*, Contemp. Math. **547** (2011), 61–71.
- [2] R. Aron and P. Rueda, *Ideals of homogeneous polynomials*, Publ. Res. Inst. Math. Sci., **46** (2012), 957–969.
- [3] R. Aron and P. Rueda, *\mathcal{I} -bounded holomorphic functions*, Preprint.
- [4] G. Botelho, *Almost summing polynomials*, Math. Nachr. **211** (2000), 25–36.
- [5] G. Botelho, *Ideals of polynomials generated by weakly compact operators*, Note Mat. **25** (2005), 69–102.
- [6] G. Botelho, H.-A. Braunss and H. Junek, *Almost p-summing polynomials and multilinear mappings*, Arch. Math. (Basel) **76** (2001), no. 2, 109–118.
- [7] G. Botelho, D. Cariello, V. Fávoro and D. Pellegrino, *Maximal spaceability in sequence spaces*, Linear Algebra Appl. **437** (2012), 2978–2985.
- [8] G. Botelho, D. Pellegrino and P. Rueda, *On Pietsch measures for summing operators and dominated polynomials*, Linear Multilinear Algebra, **62** (2014), 860–874.
- [9] G. Botelho and E.R. Torres, *Hyper-ideals of multilinear operators*, Linear Algebra Appl. **482** (2015), 1–20.
- [10] H.-A. Braunss and H. Junek, *Factorization of injective ideals by interpolation*, J. Math. Anal. Appl. **297** (2004), 740–750.
- [11] J. Campos, *Cohen and multiple Cohen strongly summing multilinear operators*, Linear Multilinear Algebra **62** (2014), no. 3, 322–346.
- [12] Y.S. Choi and J.M. Kim, *The dual space of $(\mathcal{L}(X, Y), \tau_p)$ and the p-approximation property*, J. Funct. Anal. **259** (2010), no. 9, 2437–2454.
- [13] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, Results Math. **39** (2001), 201–217.
- [14] J. Diestel, H. Jarchow, A. Tonge, *Absolutely Summing Operators*, Cambridge University Press, New York, 1995.
- [15] V. Dimant, *Strongly p-summing multilinear operators*, J. Math. Anal. Appl. **278** (2003), 182–193.
- [16] K. Floret and D. García, *On ideals of polynomials and multilinear mappings between Banach spaces*, Arch. Math. (Basel) **81** (2003), 300–308.
- [17] M. González and J.M. Gutiérrez, *Surjective factorization of holomorphic mappings*, Comment. Math. Univ. Carolin. **41** (2000), 469–476.
- [18] S. Lassalle and P. Turco, *The Banach ideal of \mathcal{A} -compact operators and related approximation properties*, J. Funct. Anal. **265** (2013), 2452–2464.
- [19] D. Pellegrino and J. Ribeiro, *On almost summing polynomials and multilinear mappings*, Linear Multilinear Algebra **60** (2012), no. 4, 397–413.
- [20] D. Pellegrino and J. Ribeiro, *On multi-ideals and polynomial ideals of Banach spaces: a new approach to coherence and compatibility*, Monatsh. Math. **173** (2014), no. 3, 379–415.
- [21] D. Pellegrino and M.L.V. Souza, *Fully and strongly almost summing multilinear mappings*, Rocky Mountain J. Math. **36** (2006), no. 2, 683–698.

- [22] A. Pietsch, *Operator Ideals*, North Holland, 1980.
- [23] A. Pietsch, *Ideals of multilinear functionals*, Proceedings of the Second International Conference on Operator Algebras, Ideals and Their Applications in Theoretical Physics, Leipzig Teubner Texte Math. **62** (1983), 185–199.
- [24] A. Pietsch, *The ideal of p -compact operators and its maximal hull*, Proc. Amer. Math. Soc. **142** (2013), no. 2, 519–530.
- [25] D. Popa, *Almost summing and multiple almost summing multilinear operators on ℓ_p spaces*, Arch. Math. (Basel) **103** (2014), no. 3, 291–300.
- [26] D. Popa, *Composition results for strongly summing and dominated multilinear operators*, Cent. Eur. J. Math. **12** (2014), 1433–1446.
- [27] D.M. Serrano-Rodríguez, *Absolutely γ -summing multilinear operators*, Linear Algebra Appl. **439** (2013), 4110–4118.
- [28] D.P. Sinha and A.K. Karn, *Compact operators whose adjoints factor through subspaces of ℓ_p* , Studia Math. **150** (2002), 17–33.
- [29] I. Stephani, *Generating systems of sets and quotients of surjective operator ideals*, Math. Nachr. **99** (1980), 13–27.

Faculdade de Matemática
 Universidade Federal de Uberlândia
 38.400-902 – Uberlândia, Brazil
 e-mails: botelho@ufu.br, ewerton@powerline.com.br.